JOURNAL OF APPROXIMATION THEORY 41, 100–113 (1984)

Degeneracy in WT-Spaces

D. Zwick

Department of Mathematics, University of Vermont, Burlington, Vermont 05405, U.S.A.

Communicated by Oved Shisha

Received March 16, 1983

Introduction

An n-dimensional linear space of functions defined on a real interval is called a WT-space (weak Tchebysheff space) if no element has more than $n-1$ sign changes. If, in addition, no nontrivial function in the space has more than $n-1$ zeros then it is a T-space. T-spaces have proved to be useful settings in which to effect the solution of numerous problems in approximation theory. In recent years an effort has been made to extend these results to families of spline functions and to WT-spaces, of which splines are an example. Evidently this effort would be facilitated by a more thorough understanding of the similarities and differences that exist between T-spaces and WT-spaces, and of the nature of WT-spaces themselves. Basic work in this area has been done by Sommer and Strauß, Stockenberg, Zalik, Zielke, and others. It is the goal of this paper to contribute to this growing body of knowledge.

Section 1 contains a brief list of definitions and propositions without proofs concerning WT- and T-spaces, and may be safely skipped by those who are already familiar with this theory. In Section 2 we introduce the notion of "degeneracy" and identify various conditions under which one may conclude that a WT-space is in fact a T-space. Section 3 deals with the zero structure of elements in WT-spaces that contain lower dimensional T-spaces, generalizing among others a theorem of Stockenberg on WT-spaces that contain a positive element. In Section 4 we consider the special case of WT-spaces with a T-space of dimension one less, and Section 5 concerns vanishing points, points at which every element of a linear space vanishes.

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In this section we record various basic facts about T-spaces and WT-spaces. Most of the material can be found in the monograph [8] of Zielke,

0021-9045/84 $3.00

where the appropriate references are also given. Readers already familiar with this theory are advised to skip to Section 2.

Let U be an n-dimensional linear space of real-valued functions defined on a subinterval of the real line. We say that an element $u \in U$ has k sign changes if there exist points $y_1 < \cdots < y_{k+1}$ such that $u(y_i)\, u(y_{i+1}) < 0$ $(i = 1,..., k)$.

DEFINITION 1.1. U is a WT-*space* iff no element has more than $n-1$ sign changes. If, in addition, no nontrivial element has more than $n-1$ zeros then U is called a T-*space*. No continuity need be assumed.

PROPOSITION 1.1. *U is a WT-space iff it has a basis $\{u_0,..., u_{n-1}\}$, a* WT-system, *such that* $\det\{u_i(x_j)\}_0^{n-1} \geqslant 0$ *for all* $x_0 < \cdots < x_{n-1}$. *If U is a WT-space then every basis for U can be made into a WT-system by reversing the sign of at most one basis element. An analogous statement holds for T-spaces and T-systems save that the above determinants are all strictly positive.*

Obviously, if U is an n-dimensional T-space then for points $x_0,..., x_{n-1}$ and for any numbers $y_0,..., y_{n-1}$ there is a unique $u \in U$ for which $u(x_i) = y_i$ $(i = 0,..., n-1)$.

PROPOSITION 2.1. *Every T-space on (a, b) or on $[a, b]$ contains a positive function.*

For a T-space on the half-open interval this may not hold.

EXAMPLE 1.1 [8, p. 43]. $U = \mathrm{sp}\{x, x^2 - 1\}$ is a WT-space on $[-1, 1]$ and a T-space on $[-1, 1)$ and $(-1, 1]$ but contains no positive function in these intervals as every element vanishes either at both endpoints or else in $(-1, 1)$.

PROPOSITION 3.1. *U is an n-dimensional T space iff no nontrivial element has more than $n-1$ zeros and sign changes counted as follows: zeros at endpoints, zeros with sign changes, and sign changes without zeros are each counted once; zeros without sign changes are counted twice.*

The following proposition is a direct consequence of Proposition 3.1.

PROPOSITION 4.1. *Let U be an n-dimensional T-space on (a, b). If $u \in U$ has $n-1$ distinct zeros and sign changes, then u has no zeros without sign changes.*

The following result, sometimes referred to as Krein's theorem [1, p. 30], has been recently revised by Zielke [8, Theorem 6.5]. We record here a simpler version, which suffices for our purposes.

PROPOSITION 5.1. *Let U be an n-dimensional T-space on $[a, b]$ and let $a < x_1 < \cdots < x_k < b$ be given with $k \leqslant n-1$. Then there is an element $u \in U$ with a sign change at each x_i, such that $u(x_i) = 0$ $(i = 1,..., k)$ and u has no other zeros in $[a, b]$.*

DEFINITION 2.1. A T- or WT-system $\{u_0,..., u_{n-1}\}$ is called *complete* if for each $k = 0,..., n-1$, $\{u_0,..., u_k\}$ is a T- or WT-system, respectively. A complete T-system is called a *CT-system.*

In particular, u_0 is positive or nonnegative, respectively.

PROPOSITION 6.1. *Any WT-space has a basis which is a complete WT-system.*

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In this section we consider the question: When is a WT-space U actually a T-space? As we shall see, the answer to this query depends on the domain of U as well as on various properties of the space, including one which we call the "T-rank"of U.

The symbol $U\binom{0,\ldots,n-1}{x_0,\ldots,x_{n-1}}$ denotes the determinant $\det\{u_i(x_j)\}_0^{n-1}$, for functions $u_0,..., u_{n-1}$ and points $x_0,..., x_{n-1}$.

DEFINITION 1.2. A function f is *adjoined* to U iff $U \cup \{f\}$ spans a T-space of dimension $\dim U + 1$. Similarly, f is adjoined to $\{u_0,..., u_{n-1}\}$ if $\{u_0,..., u_{n-1}, f\}$ is a T-system. The term *weakly adjoined* is used when $U \cup \{f\}$ spans a WT-space or $\{u_0,..., u_{n-1}, f\}$ is a WT-system.

THEOREM 1.2. *Let U be a WT-space on (a, b). If U has an adjoined function then U is a T-space on (a, b).*

Proof. Assume that U has dimension n, f is adjoined to U, and U_f denotes the resulting $(n+1)$-dimensional T-space. Suppose that $u \in U$ is nontrivial and has n zeros. Since $u \in U_f$, we may apply Proposition 4.1 and deduce that u must have n sign changes. But this contradicts our assumption that U is an n-dimensional WT-space, hence, no such element exists and U is a T-space on (a, b). ∎

Corollary 1.2 is immediate from Theorem 1.2.

COROLLARY 1.2. *If $\{u_0,..., u_{n-1}\}$ is a complete WT-system on (a, b) with an adjoined function, then it is a CT-system on (a, b).*

As another application of Theorem 1.2 we present the following result of Krein and Nudel'man [10, p. 44].

COROLLARY 2.2. *Let $\{u_0,..., u_n\}$ be a continuous T-system on $[a, b]$ and suppose there exists elements $v_0,..., v_n \in \mathrm{sp}\{u_i\}_0^n$ such that*

(1) $\lim_{t\uparrow b} v_i(t)/v_{i+1}(t) = 0$ $(i = 0,..., n-1)$ *and*

(2) $v_i(t) > 0$ *near* b $(i = 0,..., n)$.

Then $\{v_0,..., v_n\}$ is a CT-system on (a, b).

Proof. It follows from (1) that $\lim_{t\uparrow b} v_i(t)/v_j(t) = 0$ for $0 \leqslant i < j \leqslant n$, from which one easily gets the linear independence of $\{v_0,..., v_n\}$. Thus $\{\varepsilon v_0, v_1,..., v_n\}$ is a T-system on $[a, b]$ for $\varepsilon = \pm 1$. For $a \leqslant t_0 < \cdots < t_n < b$ we have

$$0 < \varepsilon V\begin{pmatrix} 0,..., n \\ t_0,..., t_n \end{pmatrix} = \varepsilon v_n(t_n) \begin{vmatrix} v_0(t_0) & \cdots & v_0(t_{n-1}) & v_0(t_n)/v_n(t_n) \\ \vdots & & \vdots & \vdots \\ v_{n-1}(t_0) & \cdots & v_{n-1}(t_{n-1}) & v_{n-1}(t_n)/v_n(t_n) \\ v_n(t_0) & \cdots & v_n(t_{n-1}) & 1 \end{vmatrix}$$

$$= \varepsilon v_n(t_n) \left| V\begin{pmatrix} 0,..., n-1 \\ t_0,..., t_{n-1} \end{pmatrix} + o(1) \right| \qquad \text{as} \quad t_n \uparrow b.$$

hence, $\varepsilon V\binom{0,...,n-1}{t_0,...,t_{n-1}} \geqslant 0$. Similarly, $\varepsilon V\binom{0,...,k}{t_0,...,t_k} \geqslant 0$ for $k = 1,..., n-2$, and $\varepsilon v_0(t_0) \geqslant 0$. Since $v_0(t) > 0$ for t near b, $\varepsilon = +1$. Moreover, $\{v_0,..., v_{n-1}\}$ is a complete WT-system on (a, b) with an adjoined function v_n, so by Corollary 1.2 $\{v_0,..., v_n\}$ is a CT-system on (a, b). ∎

In general, Theorem 1.2 is false for closed and half-open intervals, as the following simple example demonstrates.

EXAMPLE 1.2. Let $U = \mathrm{sp}\{x, x^2\}$ on $[0, 1]$ or on $[0, 1)$. The function $f(x) \equiv 1$ is adjoined to U since no nonzero element of $\mathrm{sp}\{1, x, x^2\}$ has more than two zeros, however, U is clearly not a T-space on $[0, 1]$ or on $[0, 1)$.

A well-known property of WT-spaces that distinguishes them from T-spaces is that of "degeneracy."

DEFINITION 2.2. U is said to be *degenerate* on an interval I if there exists a nontrivial element of U that vanishes identically on I. If U contains such an element we simply say that U is degenerate; otherwise U is called *nondegenerate*.

T-spaces and linear spaces spanned by analytic functions, such as polynomials, are examples of nondegenerate linear spaces.

We now return to our original question. For the open interval an answer was provided by Stockenberg [6] and by Zalik [7]; namely, a WT-space U is a T-space precisely when U is nondegenerate and contains a positive function (see also [4]). The situation is more complicated for the closed and the half-open interval.

DEFINITION 3.2. A real-valued function f is said to have k *separated zeros* if there exist points $x_1 < y_1 < x_2 < \cdots < y_{k-1} < x_k$ such that $f(x_i) = 0$ $(i = 1,..., k)$ and $f(y_i) \neq 0$ $(i = 1,..., k-1)$.

THEOREM 2.2 [6]. *Let U be an n-dimensional WT-space with a positive function. If n is even then no element has more than $n-1$ separated zeros. If n is odd then no element has more than n separated zeros and any element with n seperated zeros vanishes to the left of the first and to the right of the last zero.*

Corollary 3.2 is easily deduced from Theorem 2.2, and provides an answer to our query in all but one instance.

COROLLARY 3.2. *Let U be an n-dimensional, non-degenerate WT-space on $[a, b]$ with a positive function.*

(a) *If n is even then U is a T-space on $[a, b]$.*

(b) *If n is odd then U is a T-space on (a, b), on $(a, b]$, and on $[a, b)$, but not necessarily on $[a, b]$.*

By way of illustration, we present a counterexample of Nürnberger and Sommer [4].

EXAMPLE 2.2. Let $U = \mathrm{sp}\{1, x(1-x^2), x^2\}$. U is a 3-dimensional, nondegenerate WT-space on $[-1, 1]$ with a positive function. Although U is a T-space on $[-1, 1)$ and on $(-1, 1]$, U is clearly not a T-space on $[-1, 1]$.

Our next goal is, therefore, to determine additional conditions on U that will make it a T-space on $[a, b]$ regardless of the parity of n. Noting that a positive function generates a 1-dimensional T-space, it is natural to hypothesize that U contains a T-subspace of dimension greater than 1.

DEFINITION 4.2. A WT-space is said to have *T-rank* k if it contains a T-space of dimension k but no T-space of dimension $k+1$. T-rank 0 implies no T-subspaces.

EXAMPLE 3.2. The space $S_{n,k}(\xi_1,..., \xi_k)$ of splines of degree $n-1$ with simple knots at $\xi_1,..., \xi_k$ (see [1, p. 18]) is a WT-space of dimension $n+k$ with T-rank n, as it contains the T-space of polynomials of degree $\leqslant n-1$.

THEOREM 3.2. *Let U be a nondegenerate WT-space of dimension n on $[a, b]$. If U has T-rank $\geqslant 2$ then U is a T-space on $[a, b]$.*

Proof. Suppose that $u \in U$ is nontrivial and has n zeros, $a \leqslant x_1 < \cdots < x_n \leqslant b$. Since U has positive T-rank, Proposition 2.1 implies that U contains a positive function, say v. By Theorem 2.2 and the assumption of nondegeneracy, n must be odd, $x_1 = a$, and $x_n = b$. Let $y_1, \ldots, y_{n-1}$ be points such that $x_i < y_i < x_{i+1}$ and $u(y_i) \neq 0$ $(i = 1, \ldots, n-1)$. We claim that exactly $(n-1)/2$ of the numbers $u(y_i)$ are positive and $(n-1)/2$ of them are negative. Otherwise, for small $\lambda > 0$, $u + \lambda v$, or $u - \lambda v$ would have at least $2(n+1)/2 = n+1$ sign changes, a contradiction. Assume, without loss of generality, that $u(y_i) > 0$. Choosing a point $\xi \in (x_1, x_2)$, define an element w in the T-subspace, which satisfies $w(\xi) = 0$, $w(x) > 0$ for $x \in [a, \xi)$, and $w(x) < 0$ for $x \in (\xi, b]$. This is possible by Proposition 5.1. Then for small $\gamma > 0$, $u - \gamma w$ has n sign changes in $[a, b]$. This contradiction proves the theorem. ∎

It follows from Theorem 3.2 that, since the WT-space in Example 2.2 is not a T-space on $[-1, 1]$, it has no 2-dimensional T-subspace and, hence, has T-rank 1.

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Theorem 2.2 concerns the number of separated zeros an element of a WT-space with T-rank 1 can possess. A related result of Kimchi [2] can be paraphrased as

THEOREM 1.3. *Let U be an n-dimensional WT-space on $[a, b]$ with T-rank $n - 1$. If $u \in U$ vanishes on a subinterval of $[a, b]$ then u has but one separated zero (i.e., the set of zeros of u is an interval).*

We now proceed to generalize these two results to arbitrary T-rank.

THEOREM 2.3. *Let U be an $(n+k)$-dimensional WT-space on $[a, b]$ with T-rank n $(n, k \geqslant 1)$. Suppose that $u \in U$ is nontrivial and vanishes on a subinterval of $[a, b]$. If k is even then u has at most $k+1$ separated zeros. If k is odd then u has at most k separated zeros.*

Proof. Assume that u vanishes on $(\alpha, \beta) \subset [a, b]$. The idea of the proof will be to define a function v in the T-subspace V with $n-1$ zeros in (α, β). We then show that if u has more than the above-mentioned number of separated zeros then for some $\gamma \neq 0$, $u - \gamma v$ has $n + k$ sign changes in contradiction to our assumptions on U. We distinguish between the following two cases:

Case I (*n even*). Assume first that k is odd and u has $k+1$ separated zeros (including (α, β)), that is, there are points $\alpha \leqslant x_1 < y_1 < x_2 < \cdots < y_k < x_{k+1} \leqslant b$ such that $u(x_i) = 0$ $(i = 1,..., k+1)$ and $u(y_i) \neq 0$ $(i = 1,..., k)$. Define sets

$$A_1 = \{y_i \in [a, \alpha]: u(y_i) > 0\}, \qquad A_2 = \{y_i \in [a, \alpha]: u(y_i) < 0\},$$
$$A_3 = \{y_i \in [\beta, b]: u(y_i) < 0\}, \qquad A_4 = \{y_i \in [\beta, b]: u(y_i) > 0\}.$$

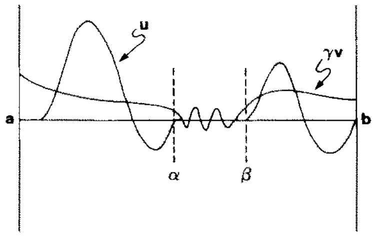
A_i $(i = 1, 2, 3, 4)$ are disjoint sets containing all of the y_i in their union. Hence, either $A_1 \cup A_3$ or $A_2 \cup A_4$ contains $(k+1)/2$ elements. We may assume the former is true. Now let $v \in V$ be defined as above with $n-1$ zeros in (α, β), all of which are sign changes, and let v be positive in $[a, \alpha]$. Since n is even, v will be negative in $[\beta, b]$. Thus, for small $\gamma > 0$, $u - \gamma v$ has at least $2(k+1)/2 = k+1$ sign changes in $[a, b] \backslash (\alpha, \beta)$, which in addition to the $n-1$ sign changes in (α, β) yields $n+k$ sign changes, the desired contradiction.

If k is even, the same argument is valid, except we must assume that n has $k+2$ separated zeros in order to derive a similar contradiction.

Case II (n odd). Again, suppose that u has $k+1$ separated zeros if k is odd, $k+2$ if k is even. Then there are at least $(k+1)/2$ positive $u(y_i)$ or negative $u(y_i)$ for k odd, $(k+2)/2$ for k even. We define $v \in V$ as in Case I, noting that v will then have the same sign in $[a, \alpha]$ as in $[\beta, b]$ in contrast to the situation in Case I. Then as before, for some $\gamma \neq 0$, $u - \gamma v$ will have at least $n+k$ sign changes for k odd or $n+k+1$ sign changes for k even, in either case a contradiction (see Figs. 1 and 2). ∎

THEOREM 3.3. *Let U be a WT-space of dimension $n+k$ on $[a, b]$ with T-rank n $(n, k \geqslant 1)$. If $u \in U$ has $k+1$ separated zeros then either all its zeros are isolated or else k is even and u vanishes at both endpoints.*

Proof. Suppose that u vanishes on $(\alpha, \beta) \subset [a, b]$. By Theorem 2.3 we may assume k is even. If n is odd then there must be precisely $k/2$ positive



(n=7 , k=4)

FIGURE 1

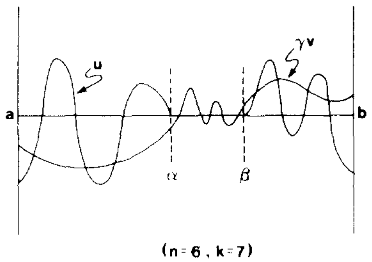


FIGURE 2

$u(y_i)$ and $k/2$ negative $u(y_i)$. Unless u vanishes on $[a, x_1]$ and on $[x_{k-1}, b]$, we may define an element $v \in V$ as in Theorem 2.3 such that for small $\gamma > 0$, $u - \gamma v$ has $n-1$ sign changes in (α, β) and $2(k/2)+1 = k+1$ sign changes in $[a, b] \backslash [\alpha, \beta]$, a contradiction.

For n even the situation is similar except that in this case the sets $A_1 \cup A_3$ and $A_2 \cup A_4$ of Theorem 2.3 must each contain precisely $k/2$ of the y_i. ∎

Note that Theorem 2.2 and Theorem 1.3 are contained in Theorems 2.3 and 3.3 as the special cases $n = 1$ and $k = 1$, respectively.

4

This section is devoted to displaying some of the phenomena that arise when a T-system is extended by an additional function to a WT-system. Although expressed in terms of systems, one could reformulate these assertions in terms of T-spaces of dimension n with T-rank $n-1$, as in Theorem 1.3. We begin with a result of Lapidot.

THEOREM 1.4 [3]. *Let $\{u_0, ..., u_{n-1}\}$ be a T-system on $[a, b]$ and assume that $\{u_0, ..., u_{n-1}, f\}$ is a WT-system on $[a, b]$. If for some $a \leqslant x_0 < \cdots < x_n \leqslant b$,*

$$U\begin{pmatrix} 0, ..., n-1; f \\ x_0, ..., x_n \end{pmatrix} = \begin{vmatrix} u_0(x_0) & \cdots & u_0(x_n) \\ \vdots & & \vdots \\ u_{n-1}(x_0) & \cdots & u_{n-1}(x_n) \\ f(x_0) & \cdots & f(x_n) \end{vmatrix} = 0,$$

then there exists a unique $u \in \mathrm{sp}\{u_i\}_0^{n-1}$ such that $f \equiv u$ on $[x_0, x_n]$.

Theorem 1.4 has the following interesting corollary.

COROLLARY 1.4. *If $\{u_0, ..., u_{n-1}\}$ is a T-system and $\{u_0, ..., u_{n-1}, f\}$ is a WT-system on $[a, b]$ then $U_f = \mathrm{sp}\{u_0, ..., u_{n-1}, f\}$ is either degenerate or a T-system on $[a, b]$.*

Proof. Either for all $a \leqslant x_0 < \cdots < x_n \leqslant b$, $U\binom{0,\ldots,n-1;f}{x_0,\ldots,x_n} > 0$ or else, by Theorem 1.4, U_f is degenerate. ∎

We note further that, under these conditions, for all $a < x_1 < \cdots < x_{n-1} < b$, $U\binom{0,\ldots,n-1;f}{a,x_1,\ldots,x_{n-1},b} > 0$, for otherwise Theorem 1.4 implies that $f \in \mathrm{sp}\{u_0,\ldots,u_{n-1}\}$, in contradiction to the assumption that $\{u_0,\ldots,u_{n-1},f\}$ is a basis, hence linearly independent.

COROLLARY 2.4. *Let $\{u_0,\ldots,u_{n-1}\}$ be a complete WT-system with $u_0 > 0$.*

(a) *If $\mathrm{sp}\{u_i\}_0^{n-1}$ is nondegenerate then $\{u_0,\ldots,u_{n-1}\}$ is a complete T-system.*

(b) *If $\mathrm{sp}\{u_i\}_0^{n-1}$ has an adjoined function then $\{u_0,\ldots,u_{n-1}\}$ is a complete T-system.*

Proof. (a) Since $u_0 > 0$ and $\mathrm{sp}\{u_0,u_1\}$ is nondegenerate, by Corollary 1.4, $\{u_0,u_1\}$ is a T-system. This reasoning may be applied successively to $\{u_0,\ldots,u_j\}$ $(j = 1,\ldots,n-1)$ to get the desired result.

(b) If $\mathrm{sp}\{u_i\}_0^{n-1}$ has an adjoined function then it must be nondegenerate, hence, part (b) follows from part (a). ∎

Combining Theorem 1.3 and Corollary 1.4 we get Theorem 2.4.

THEOREM 2.4. *If $\{u_0,\ldots,u_{n-1}\}$ is a T-system and $\{u_0,\ldots,u_{n-1},f\}$ is a WT-system on $[a,b]$ then either $\{u_0,\ldots,u_{n-1},f\}$ is a T-system on $[a,b]$ or else for some $u \in \mathrm{sp}\{u_i\}_0^{n-1}$, $f-u$ vanishes on a subinterval $[\alpha,\beta) \subset [a,b]$ and is nonzero in $[a,b]\backslash[\alpha,\beta]$. If the latter holds then $f-u > 0$ in $(\beta,b]$ and $(-1)^n(f-u) > 0$ in $[a,\alpha)$.*

Proof. If $\{u_0,\ldots,u_{n-1},f\}$ is not a T-system then for some $u \in \mathrm{sp}\{u_i\}_0^{n-1}$, $f \equiv u$ on a subinterval of $[a,b]$. Let $\alpha = \inf\{x: f(x) = u(x)\}$, $\beta = \sup\{x: f(x) = u(x)\}$, then by Theorem 1.3, $f \equiv u$ on (α,β), and $f-u$ is nonzero in $[a,b]\backslash[\alpha,\beta]$. The above inequalities follow from the identity $f(x) - u(x) = U\binom{0,\ldots,n-1;f}{x_0,\ldots,x_{n-1},x} / U\binom{0,\ldots,n-1}{x_0,\ldots,x_{n-1}}$ for $\alpha < x_0 < \cdots < x_{n-1} < \beta$. ∎

5

We start this section with a definition.

DEFINITION 1.5. A *vanishing point* for a linear space U is a point ξ such that $u(\xi) = 0$ for all $u \in U$.

Clearly T-spaces can have no vanishing points. If, however, one multiplies each element of a T-space by a fixed nonnegative function ω, then the result is a WT-space for which the zeros of ω are the vanishing points. If U is a nondegenerate WT-space then this process may be reversed, as we will see later.

If U contains a positive function, or indeed one that is nowhere zero, then U has no vanishing points. Under certain conditions the converse is also true, as we show in Theorems 2.5 and Theorem 3.5.

Our first theorem of this section characterizes vanishing points.

THEOREM 1.5. *Let* $\{u_0,..., u_{n-1}\}$ *be linearly independent on* $[a, b]$. *Then* $\xi \in [a, b]$ *is a vanishing point for* $U = \mathrm{sp}\{u_i\}_0^{n-1}$ *iff, for some set of points* $a \leqslant x_0 < \cdots < x_{n-1} \leqslant b$ *such that*

$$U\begin{pmatrix} 0,..., n-1 \\ x_0,..., x_{n-1} \end{pmatrix} \neq 0, \qquad U\begin{pmatrix} 0,..., n-1 \\ \xi, x_0,..., x_{j-1}, x_{j+1},..., x_{n-1} \end{pmatrix} = 0$$

$(j = 0,..., n-1)$.

Proof. One direction is obvious; the converse is proved by induction. As the case $n = 1$ is trivial, assume validity of the theorem for all sets of $n-1$ functions. Let $y_1,..., y_{n-2}$ be distinct points in $\{x_0,..., x_{n-1}\}$, then by assumption

$$0 = U\begin{pmatrix} 0,..., n-1 \\ \xi, y_1,..., y_{n-2}, x_j \end{pmatrix} = \sum_{i=0}^{n-1} a_i u_i(x_j) \qquad (j = 0,..., n-1),$$

where

$$a_i = (-1)^{n-1-i} U\begin{pmatrix} 0,..., i-1, i+1,..., n-1 \\ \xi, y_1,..., y_{n-2} \end{pmatrix}.$$

Since the determinant of this linear system, $\det\{u_i(x_j)\}_0^{n-1}$, is nonzero, it follows that $a_i = 0$ $(i = 0,..., n-1)$. For each $0 \leqslant i \leqslant n-1$ $\{u_0,..., u_{i-1}, u_{i+1},..., u_{n-1}\}$ is linearly independent on $\{x_0,..., x_{n-1}\}$, hence, there is a subset $\{y_0^i,..., y_{n-2}^i\}$ for which

$$U\begin{pmatrix} 0,..., i-1, i+1,..., n-1 \\ y_0^i,..., y_{n-2}^i \end{pmatrix} \neq 0.$$

Thus the inductive hypothesis applies and we may conclude that ξ is a vanishing point for $\mathrm{sp}\{u_0,..., u_{i-1}, u_{i+1},..., u_{n-1}\}$ $(i = 0,..., n-1)$ and, consequently, for U as well. ∎

Since $\{u_0,..., u_{n-1}\}$ is linearly independent iff points $x_0,..., x_{n-1}$ exist such that $U\binom{0,...,n-1}{x_0,...,x_{n-1}} \neq 0$, it follows that ξ is a vanishing point iff

$U\binom{0,\ldots,n-1}{\xi, y_1,\ldots,y_{n-1}}=0$ for all $y_1,\ldots,y_{n-1}$ in any set on which $\{u_0,\ldots,u_{n-1}\}$ is linearly independent.

The next theorem says that if, for a fixed, finite set of points Y, every function in a linear space has a zero in Y, then Y must contain a vanishing point.

THEOREM 2.5 (cf. [6, Lemma 2]). *Let F be a linear space of functions on a set X. Then for any finite subset $Y \subset X$ with no vanishing points there is an element of F that is nonzero on Y.*

Proof. Assume that $Y=\{y_1,\ldots,y_k\}$ contains no vanishing points. The proof is by induction on k. For $k=1$ there is nothing to prove, so assume validity of the theorem for sets of $k-1$ or fewer points. Then, since $\{y_1,\ldots,y_{k-1}\}$ contains no vanishing points, there is a $g \in F$ such that $g(y_i) \neq 0$ $(i=1,\ldots,k-1)$. If $g(y_k) \neq 0$ then we are done; otherwise, let $f \in F$ be a function for which $f(y_k) \neq 0$. Such a function exists as y_k is not a vanishing point. Choose a constant $\alpha \neq 0$ such that

$$\max_{1 \leqslant i \leqslant k-1} |\alpha f(y_i)| < \min_{1 \leqslant i \leqslant k-1} |g(y_i)|,$$

then $(\alpha f+g)(y_i) \neq 0$ $(i=1,\ldots,k-1)$ and $(\alpha f+g)(y_k)=\alpha f(y_k) \neq 0$. Hence, $\alpha f+g$ is nonzero on Y and the theorem is proved. ∎

Theorem 2.5 can be used as in [6] to prove a result similar to Theorem 2.2, that if U is an n-dimensional WT-space then any element with n separated zeros, none of which are at vanishing points, vanishes to the right of the last and to the left of the first of these zeros. It then follows that a nondegenerate WT-space on (a, b) with no vanishing points is a T-space. Since both imply that a WT-space is a T-space, one may deduce that for a nondegenerate WT-space, having no vanishing points is equivalent to having a positive function. This was pointed out by Zalik [7]. For a degenerate space this may be false, as the next example demonstrates.

EXAMPLE 1.5. Let $u_0(x)=x$ and

$$\begin{aligned} u_1(x) &= 1-|x|, & x &\in [-1,1], \\ &= 0, & x &\in (-2,-1) \cup (1,2). \end{aligned}$$

It is evident that for all $\gamma \neq 0$, γu_0 intersects u_1 at most once. Hence, $U=\operatorname{sp}\{u_0, u_1\}$ is a WT-space on $(-2,2)$; U has no vanishing points, however, it contains no positive element since every function $u \in U$ satisfies $\operatorname{sgn} u_{|(-2,-1)} \cdot \operatorname{sgn} u_{|(1,2)} \leqslant 0$.

It turns out that such a counterexample as Example 1.5 is generic: Every WT-space on (a, b) with neither vanishing points nor a positive element contains a function that vanishes on an interval that extends to an endpoint. This is the content of Theorem 3.5, which is proved with the aid of the next lemma.

LEMMA 1.5. *Let* $\{u_0, \ldots, u_{n-1}\}$ *be a WT-system on* (a, b) *and let* $\xi \in (a, b)$. *If* $U = \mathrm{sp}\{u_i\}_0^{n-1}$ *is degenerate neither on* (a, ξ) *nor on* (ξ, b) *then there exist points*

$$a < x_0 < \cdots < x_{n-2} < \xi < x_{n-1} < b$$

such that $U\binom{0, \ldots, n-1}{x_0, \ldots, x_{n-1}} > 0$.

Proof. Since U is not degenerate on (a, ξ) there must be points $a < x_0 < \cdots < x_{n-2} < \hat{x} < \xi$ such that $U\binom{0, \ldots, n-1}{x_0, \ldots, x_{n-2}, \hat{x}} > 0$. Let $u(x) = U\binom{0, \ldots, n-1}{x_0, \ldots, x_{n-2}, x}$, then u is a nontrivial element of U since $u(\hat{x}) \neq 0$. Moreover, u cannot venish identically in (ξ, b) for then U would be degenerate on (ξ, b). Hence, there is a point x_{n-1} in (ξ, b) such that $u(x_{n-1}) \neq 0$. It follows that $U\binom{0, \ldots, n-1}{x_0, \ldots, x_{n-1}} > 0$ and so $x_0, \ldots, x_{n-1}$ are the desired points. ∎

Lemma 1.5 figures prominently in [9], where the author shows, for example, that for a continuous WT-space a weakly adjoined function is continuous at every point ξ that satisfies the conditions of the lemma.

THEOREM 3.5. *Let U be a WT*-space *on* (a, b) *such that for each* $\xi \in (a, b)$ *U is degenerate neither on* (a, ξ) *nor on* (ξ, b). *Then either U contains a positive function or else U has a vanishing point in* (a, b).

Proof. Let $\{u_0, \ldots, u_{n-1}\}$ be a basis for U which is a complete WT-system (Proposition 6.1). If $u_0 > 0$ then we are done, so assume that $u_0(\xi) = 0$ for some $\xi \in (a, b)$. We will show that ξ is a vanishing point for U. Let k be the largest integer such that $u_0(\xi) = \cdots = a_{k-1}(\xi) = 0$ and suppose that $k < n$. Since U is not degenerate on (a, ξ) there are points $a < x_0 < \cdots < x_{k-1} < \xi$ such that $U\binom{0, \ldots, k-1}{x_0, \ldots, x_{k-1}} > 0$. Hence,

$$U\begin{pmatrix} 0, \ldots, k \\ x_0, \ldots, x_{k-1}, \xi \end{pmatrix} = u_k(\xi)\, U\begin{pmatrix} 0, \ldots, k-1 \\ x_0, \ldots, x_{k-1} \end{pmatrix},$$

from which we conclude that $u_k(\xi) \geqslant 0$. By Lemma 1.5 there exist points $a < t_0 < \cdots < t_{k-2} < \xi < t_{k-1} < b$ for which $U\binom{0, \ldots, k-1}{t_0, \ldots, t_{k-1}} > 0$. For these points

$$U\begin{pmatrix} 0, \ldots, k \\ t_0, \ldots, t_{k-1}, y_k \end{pmatrix} = u_k(\xi)\, U\begin{pmatrix} 0, \ldots, k-1 \\ t_0, \ldots, t_{k-1} \end{pmatrix},$$

yielding $u_k(\xi) \leqslant 0$. Thus, $u_k(\xi) = 0$, contradicting the maximality of k, unless $k = n$. This proves the theorem. ∎

It is interesting to note that under the conditions of Theorem 3.5, no element has more than $n-1$ separated zeros, ignoring possible vanishing points. Example 1.1 shows that Theorem 3.5 is not generally true when applied to closed or half-open intervals. Corollary 1.5 follows immediately from the proof of Theorem 3.5.

COROLLARY 1.5. *Let U be a WT-space on (a, b) and assume that no element vanishes on an interval that extends to an endpoint.*

(a) *If U has a vanishing point $\xi \in (a, b)$ then every weakly adjoined function vanishes at ξ.*

(b) *If $\{u_0,..., u_{n-1}\}$ is any complete WT-system spanning U, then the vanishing points for U are precisely the zeros of u_0.*

Corollary 1.5 is especially important in light of Proposition 6.1. Moreover, it may be used to provide another proof of the following result.

COROLLARY 2.5 [8, p. 32]. *Every T-system on an open interval has a basis which is a complete T-system.*

Proof. If U is a T-space on (a, b) then U is, in particular, a WT-space, so it has a basis which is a complete WT-system, say $\{u_0,..., u_{n-1}\}$. Since U is nondegenerate and has no vanishing points it follows from Corollary 1.5b that $u_0 > 0$. The assertion now follows from Corollary 2.4a. ∎

Corollary 2.5 is not true for half-open or closed intervals. Indeed, the space in Example 1.1 is a T-space on $[-1, 1)$ with no positive element and, therefore, no basis that it is a complete T-system. Additional counterexamples, for the closed interval as well, can be found in [8, Chapter 10]. We end with a further application of the preceding results.

THEOREM 4.5. *Let $\{u_0,..., u_{n-1}\}$ be WT-system on (a, b) with nondegenerate linear span. Then there exists a nonnegative function ω, and functions $v_0,..., v_{n-1}$ such that $u_i = \omega v_i$ $(i = 0,..., n-1)$ and $\{v_0,..., v_{n-1}\}$ is a T-system on (a, b).*

In other words, $\{u_0,..., u_{n-1}\}$ may be "factored" into a T-system and a nonnegative function whose zeros are the vanishing points of $\mathrm{sp}\{u_i\}_0^{n-1}$. We now sketch the proof of Theorem 4.5:

By Proposition 6.1 we may assume that $\{u_0,..., u_{n-1}\}$ is a complete WT-system. Since $\mathrm{sp}\{u_i\}_0^{n-1}$ is nondegenerate, the set $A = \{x \in (a, b): u_0(x) > 0\}$ is dense in (a, b). As in Corollary 2.4a, $\{u_0,..., u_{n-1}\}$ is a complete T-system on A, hence, $\{1, v_1,..., v_{n-1}\}$ is too, where $v_i(x) = u_i(x)/u_0(x)$ $(i = 1,..., n-1)$, $x \in A$. Due to the presence of the function 1, the v_i have bounded variation and are bounded on closed subsets of $A \cap (a, b)$. Hence, right- and left-hand limits exist in A, so the v_i may be extended in an obvious way to the rest of (a, b). For proofs of these statements the reader is referred to [8, especially Chaps. 8, 11, 14]. We note that, once this factorization has been effected, we may produce a T-system by redefining ω to be positive at each of its zeros.

ACKNOWLEDGMENT

The material in this paper is primarily extracted from my dissertation. I am indebted to my thesis advisor, Professor H. L. Loeb, for his sound advice and invaluable support throughout my graduate years at the University of Oregon.

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